ISSN 1064–5624, Doklady Mathematics, 2006, Vol. 73, No. 1, pp. 6–10. © Pleiades Publishing, Inc., 2006.
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MATHEMATICS

A Cauchy-Type Problem for the Diffusion-Wave Equation with Riemann–Liouville Partial Derivative

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Presented by Academician S.M. Nikol'skii April 13, 2005

Received May 26, 2005

DOI: 10.1134/S1064562406010029

Consider the linear differential equation

$$(D^{\alpha}_{0+,t}u)(x,t) = \lambda^2\Delta_x u(x,t), \quad x \in \mathbf{R}^n, \quad t > 0, \quad (1)$$

where $\lambda > 0$; $(D^{\alpha}_{0+,t}u)(x,t)$ is the partial Riemann–Liouville fractional derivative of order $\alpha > 0$ of the function $u(x, t)$ with respect to the second variable [1, p. 342], i.e.,

$$(D^{\alpha}_{0+,t}u)(x,t) = \left(\frac{\partial}{\partial t}\right)^n \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(x,\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \quad (2)$$

$$\alpha > 0; \quad n = [\alpha]+1; \quad x \in \mathbf{R}^m; \quad t > 0,$$

and Δ_x is the Laplace operator in the first variable $x = (x_1, x_2, \ldots, x_m) \in \mathbf{R}^m$, i.e., $\Delta_x = \sum_{j=1}^{m}\frac{\partial^2 u}{\partial x_j^2}$.

If $\alpha = 1$ and $\alpha = 2$, then $(D^{1}_{0+,t}u)(x,t) = \frac{\partial u(x,t)}{\partial t}$ and $(D^{2}_{0+,t}u)(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}$, respectively; thus, for $\alpha = 1$, Eq. (1) coincides with the heat (diffusion) equation

$$\frac{\partial u}{\partial t} = \lambda^2\Delta_x u(x,t), \quad x \in \mathbf{R}^n, \quad t > 0, \quad (3)$$

and for $\alpha = 2$, it coincides with the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \lambda^2\Delta_x u(x,t), \quad x \in \mathbf{R}^n, \quad t > 0. \quad (4)$$

For this reason, Eq. (1) is called the diffusion-wave equation [2, p. 146]. In particular, for $m = 1$, Eq. (1) takes the form

$$(D^{\alpha}_{0+,t}u)(x,t) = \lambda^2\frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbf{R}, \quad t > 0; \quad \lambda > 0. \quad (5)$$

The interest to Eqs. (1) and (5) is caused by their numerous applications to solving diffusion problems of physics, mechanics, and other applied sciences [2, Sections 4.2.1 and 4.2.2; 3; 4] (historical information and a review of the related results are contained in survey [5, Ch. 7]).

In this paper, we solve Eq. (1) of order $\alpha > 0$ with initial conditions

$$(D^{\alpha-k}_{0+,t}u)(x,0+) = f_k(x), \quad (6)$$

$$k = 1, 2, \ldots, n = -[-\alpha]; \quad x \in \mathbf{R}^m.$$

Here, $(D^{\alpha-k}_{0+,t}u)(x,0+)$ with $n-1 < \alpha < n$ is understood as

$$(D^{\alpha-k}_{0+,t}u)(x,0+) = \lim_{t\to 0+}(D^{\alpha-k}_{0+,t}u)(x,t), \quad (7)$$

$$k = 1, 2, \ldots, n-1,$$

$$(D^{\alpha-n}_{0+,t}u)(x,0+) = \lim_{t\to 0+}(I^{n-\alpha}_{0+,t}u)(x,t); \quad (8)$$

for $\alpha = n \in N$, it equals

$$(D^{n-k}_{0+,t}u)(x,0+) = \frac{\partial^{n-k}}{\partial t^{n-k}}u(x,0), \quad (9)$$

$$k = 1, 2, \ldots, n,$$

where $(I^{n-\alpha}_{0+,t}u)(x,t)$ is the partial Riemann–Liouville fractional integral of order $n - \alpha$ [1, p. 341], that is,

$$(I^{n-\alpha}_{0+,t}u)(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(x,\tau)d\tau}{(t-\tau)^{1-n+\alpha}}, \quad \alpha < n, \quad (10)$$

$$(I^{0}_{0+,t}u)(x,t) = u(x,t).$$

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If $\alpha = n \in N$, then, according to (2) and (9), problem (1), (6) takes the form of the Cauchy problem

$$\frac{\partial^n u(x,t)}{\partial t^n} = \lambda^2 \Delta_x u(x,t), \quad x \in \mathbf{R}^m, \quad t > 0, \qquad (11)$$

$$\frac{\partial^{n-k} u(x,0)}{\partial t^{n-k}} = f_k(x), \quad k = 1, 2, \ldots, n; \; x \in \mathbf{R}^m. \qquad (12)$$

For this reason, by analogy, problem (1), (6) is called a Cauchy-type problem.

To solve problem (1), (6), we apply, respectively, the Laplace and Fourier transforms of the function $u(x, t)$ with respect to $t > 0$ and $x \in \mathbf{R}^m$, which are

$$(L_t u)(x,s) = \int_0^\infty u(x,t) e^{-st} dt, \quad x \in \mathbf{R}^m, \quad s \in C, \qquad (13)$$

$$(F_x u)(\sigma,t) = \int_{\mathbf{R}^m} u(x,t) e^{ix\cdot\sigma} dx, \quad \sigma \in \mathbf{R}^m, \quad t > 0, \qquad (14)$$

and their inverse transforms with respect to $s \in C$ and $\sigma \in \mathbf{R}^m$:

$$(L_s^{-1} u)(x,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} u(x,s) ds, \qquad x \in \mathbf{R}^m, \quad t > 0, \qquad (15)$$

$$(F_\sigma^{-1} u)(x,t) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} u(\sigma,t) e^{-i\sigma\cdot x} d\sigma, \qquad x \in \mathbf{R}^m, \quad t > 0. \qquad (16)$$

Here, $x \cdot \sigma = \sum_{i=1}^{m} x_i \sigma_i$ for $x = (x_1, x_2, \ldots, x_m) \in \mathbf{R}^m$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbf{R}^m$; $\gamma \in \mathbf{R}$ is a fixed real number. The properties of the direct and inverse Laplace and Fourier transforms can be found in, e.g., [6, Chapter II; 7, Chapter 16]. In particular, the operators in (15), (13) and (16), (14) are mutually invertible for sufficiently good functions $u(x, t)$.

Suppose that, for the functions $f_k(x)$ ($k = 1, 2, \ldots, n$) in (6), the Fourier transforms $(F_x f_k)(\sigma)$ are defined. Applying Laplace transform (13) to both sides of Eq. (1) and taking into account the expression for the Laplace transform of partial Riemann–Liouville fractional derivative (2) [2, (2.248)] and initial conditions (6), we obtain

$$s^\alpha (L_t u)(x,s) - \sum_{k=1}^{n} s^{k-1} f_k(x) = \lambda^2 \Delta_x (L_t u)(x,s).$$

Applying Fourier transform (14) to this equality and using the expression for the Fourier transform of the operator Δ_x: $(F_x \Delta_x u)(\sigma, s) = -|\sigma|^2 (F_x u)(\sigma, s)$, we come to the relation

$$(F_x L_t u)(\sigma,s) = \sum_{k=1}^{n} \frac{s^{k-1}}{s^\alpha + \lambda^2 |\sigma|^2} (F_x f_k)(\sigma), \qquad \sigma \in \mathbf{R}^m, \quad s \in C, \qquad (17)$$

where $|\sigma|^2 = \sum_{i=1}^{m} \sigma_i^2$. Applying inverse Laplace transform (15) and Fourier transform (16), we obtain a solution $u(x, t)$ to initial problem (1), (6); this is

$$u(x,t) = \left(L_s^{-1} F_\sigma^{-1} \left[\sum_{k=1}^{n} \frac{s^{k-1}}{s^\alpha + \lambda^2 |\sigma|^2} (F_x f_k)(\sigma) \right] \right)(x,t). \qquad (18)$$

Let us express solution (18) in terms of the Mittag-Leffler special function [8, 18.1(18)]

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad z \in C, \qquad (19)$$

which is an entire function of z. The formula for the Laplace transform of this function (see, e.g., [1, (1.93)]) implies

$$(L_t[t^{\alpha-k} E_{\alpha,\alpha-k+1}(-\lambda^2 |\sigma|^2 t^\alpha)])(s) = \frac{s^{k-1}}{s^\alpha + \lambda^2 |\sigma|^2}, \qquad (20)$$

$$k = 1, 2, \ldots, n; \; s \in C, \; \sigma \in \mathbf{R}^m, \; \lambda > 0; \; \lambda|\sigma||s|^{-2/\alpha} < 1.$$

Applying the inverse Laplace and Fourier transforms to both sides of (17) and taking into account (20), we obtain a solution to problem (1), (6) in the form

$$u(x,t) = \sum_{k=1}^{n} \frac{t^{\alpha-k}}{(2\pi)^m} \int_{\mathbf{R}^m} E_{\alpha,\alpha-k+1} \times (-\lambda^2 |\sigma|^2 t^\alpha) (F_x f_k)(\sigma) e^{-ix\cdot\sigma} d\sigma. \qquad (21)$$

Theorem 1. *Suppose that, for the functions $f_k(x)$ with $k = 1, 2, \ldots, n$, the Fourier transforms $(F_x f_k)(\sigma)$ are defined and the integrals on the right-hand side of* (21) *converge.*

Then, Cauchy-type problem (1), (6) *is solvable, and its solution is given by* (21).

In particular, for $m = 1$, equality (21) *gives a solution to problem* (5), (6).

Corollary 1. *The solution to Cauchy problem* (11), (12) *is given by formula* (21) *with $\alpha = n$ provided that the integrals on the right-hand side of* (21) *converge.*

If the functions $f_k(x)$ with $k = 1, 2, \ldots, n$ are infinitely differentiable on $\mathbf{R}^m$, then, substituting (19) into (21), interchanging integration and summation [which can be done because series (19) uniformly converges] and taking into account the equality

$$(\Delta^j f_k)(x) = \frac{1}{(2\pi)^m}\int_{\mathbf{R}^m}(-|\sigma|^2)^j (F_x f_k)(\sigma)e^{-i\sigma\cdot x}d\sigma,$$
$$j = 1, 2, \ldots,$$

for the jth powers of the Laplace operator ($\Delta_x^1 = \Delta_x$ and $\Delta_x^j = \Delta_x^1\Delta_x^{j-1}$ for $j = 2, 3, \ldots$), we obtain the representation of solution (21) in the form

$$u(x, t) = \sum_{k=1}^{n} t^{\alpha-k}\sum_{j=0}^{\infty}\frac{(\lambda^2 t^{\alpha})^j}{\Gamma(\alpha j+\alpha-k+1)}(\Delta_x^j f_x)(x). \quad (22)$$

Theorem 2. *Suppose that the functions $f_k(x)$, where $k = 1, 2, \ldots, n$, are infinitely differentiable on $\mathbf{R}^m$ and the series on the right-hand side of* (22) *converge*.

Then, Cauchy-type problem (1), (6) *is solvable, and its solution is given by formula* (22).

In particular, for $m = 1$, the solution to problem (5), (6) *has the form*

$$u(x, t) = \sum_{k=1}^{n} t^{\alpha-k}\sum_{j=0}^{\infty}\frac{(\lambda^2 t^{\alpha})^j}{\Gamma(\alpha j+\alpha-k+1)}f_k^{(2j)}(x). \quad (23)$$

Corollary 2. *The solution to Cauchy problem* (11), (12) *is given by formula* (22) *with $\alpha = n$ provided that the series on the right-hand side of* (22) *converge.*

If $0 < \alpha < 2$, then the solution to Eq. (1) with initial conditions (6) for $0 < \alpha \le 1$ and $n = 1$ or $1 < \alpha < 2$ and $n = 2$ can be expressed in terms of the so-called H-function [9, Section 8.3; 10, Chapters 1 and 2]. The proof is based on the application of the inverse Fourier and Laplace transforms to (17) and on the expressions for the Fourier and Laplace transforms of the modified Bessel function $K_\nu(z)$ of the third kind [11, 7.2(13)] and of the special H-function

$$H_{2,2}^{2,0}\left[z\left|\begin{matrix}\left(a, \frac{1}{2}\right), \left(b, \frac{\alpha}{2}\right)\\ (c, 1), \left(d, \frac{1}{2}\right)\end{matrix}\right.\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\Gamma(c+\tau)\Gamma(d+\tau/2)}{\Gamma(a+\tau/2)\Gamma(b+\alpha\tau/2)}z^{-\tau}d\tau. \quad (24)$$

Here, $a, b, c, d \in \mathbf{R}$ and L is a special infinite contour on the right of the poles of the gamma-functions $\Gamma(c+\tau)$ and $\Gamma(d+\tau/2)$. Theorem 1.1 from [10] implies the existence of the function defined by (24) for $0 < \alpha < 2$; $a, b, c, d \in \mathbf{R}$; and $z \neq 0$ with $|\arg(z)| < (2-\alpha)\frac{\pi}{4}$.

Lemma 1. *For $m \in N$ and $c > 0$,*

$$\left(F_x\left[|x|^{\frac{2-m}{2}}K_{\frac{m-2}{2}}(c|x|)\right]\right)(\sigma) = \left(\frac{2\pi}{c}\right)^{\frac{m}{2}}\frac{c}{c^2+|\sigma|^2}, \quad (25)$$
$$\sigma \in \mathbf{R}^m.$$

According to (25) with $c = \frac{s^{\alpha/2}}{\lambda}$ and the theorem about the Fourier convolution, relation (17) takes the form

$$(F_x L_t u)(\sigma, s)$$
$$= \left(F_x\left[\sum_{k=1}^{n}\frac{s^{k-1+\frac{\alpha(m-2)}{4}}}{\lambda(2\lambda\pi)^{\frac{m}{2}}}|x|^{\frac{2-m}{2}}K_{\frac{m-2}{2}}\left(\frac{|x|}{\lambda}s^{\frac{\alpha}{2}}\right)*_x f_k(x)\right]\right)(\sigma).$$

Applying inverse Fourier transform (16) to this equality, we obtain

$$(L_t u)(x, s)$$
$$= \sum_{k=1}^{n}\frac{s^{k-1+\frac{\alpha(m-2)}{4}}}{\lambda(2\lambda\pi)^{\frac{m}{2}}}|x|^{\frac{2-m}{2}}K_{\frac{m-2}{2}}\left(\frac{|x|}{\lambda}s^{\frac{\alpha}{2}}\right)*_x f_k(x) \quad (26)$$

for $x \in \mathbf{R}^m$ and $s \in C$.

Lemma 2. *If $0 < \alpha \le 1$ and $k = 1$ or $1 < \alpha < 2$ and $k = 1, 2$, then*

$$\left(L_t\left[t^{-k-\frac{\alpha(m-2)}{4}}H_{2,2}^{2,0}\left[\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}\left|\begin{matrix}\left(\frac{m}{4}, \frac{1}{2}\right), & \left(1-k-\frac{\alpha(m-2)}{4}, \frac{\alpha}{2}\right)\\ \left(\frac{m}{2}-1, 1\right) & \left(\frac{1}{2}-\frac{m}{4}, \frac{1}{2}\right)\end{matrix}\right.\right]\right]\right)(s)$$
$$= 2^{\frac{m}{2}}\pi^{-1/2}s^{k-1+\frac{\alpha(m-2)}{4}}K_{\frac{m-2}{2}}\left(\frac{|x|}{\lambda}s^{\frac{\alpha}{2}}\right). \quad (27)$$

Applying inverse Laplace transform (15) to (26) and taking into account (27), we obtain the following solution to problem (1), (6):

$$u(x,t) = \frac{2^{-m}|x|^{\frac{2-m}{2}}}{\lambda^{1+\frac{m}{2}}\pi^{\frac{m-1}{2}}}\sum_{k=1}^{n} t^{-k-\frac{\alpha(m-2)}{4}}$$

$$\times H^{2,0}_{2,2}\left[\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}\middle|\begin{matrix}\left(\frac{m}{4},\frac{1}{2}\right), & \left(1-k-\frac{\alpha(m-2)}{4},\frac{\alpha}{2}\right)\\ \left(\frac{m}{2}-1,1\right) & \left(\frac{1}{2}-\frac{m}{4},\frac{1}{2}\right)\end{matrix}\right] *_x f_k(x), \quad (28)$$

where $n = 1$ for $0 < \alpha \le 1$ and $n = 2$ for $1 < \alpha < 2$. This and the Fourier convolution imply the following results.

Theorem 3. *If* $0 < \alpha \le 1$, $m \in N$, *and* $\lambda > 0$, *then the Cauchy-type problem*

$$\begin{aligned}(D^{\alpha}_{0+,t}u)(x,t) &= \lambda^2(\Delta_x u)(x,t),\\ (D^{\alpha-1}_{0+,t}u)(x,0+) &= d(x), \quad x \in \mathbf{R}^m\end{aligned} \quad (29)$$

is solvable, and its solution has the form

$$u(x,t) = \int_{\mathbf{R}^m} G_1^{\alpha}(x-\tau,t)f(\tau)d\tau, \quad (30)$$

$$G_1^{\alpha}(x,t) = \frac{2^{-m}|x|^{\frac{2-m}{2}}}{\lambda^{1+\frac{m}{2}}\pi^{\frac{m-1}{2}}} t^{-1-\frac{\alpha(m-2)}{4}}$$

$$\times H^{2,0}_{2,2}\left[\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}\middle|\begin{matrix}\left(\frac{m}{4},\frac{1}{2}\right), & \left(-\frac{\alpha(m-2)}{4},\frac{\alpha}{2}\right)\\ \left(\frac{m}{2}-1,1\right) & \left(\frac{1}{2}-\frac{m}{4},\frac{1}{2}\right)\end{matrix}\right], \quad (31)$$

provided that the integral in (30) *converges.*

Theorem 4. *If* $1 < \alpha < 2$, $m \in N$, *and* $\lambda > 0$, *then the Cauchy-type problem for Eq.* (1) *with initial conditions*

$$\begin{aligned}(D^{\alpha-1}_{0+,t}u)(x,0+) &= f_1(x),\\ (D^{\alpha-2}_{0+,t}u)(x,0+) &= f_2(x), \quad x \in \mathbf{R}^m\end{aligned} \quad (32)$$

is solvable, and its solution has the form

$$u(x,t) = \int_{\mathbf{R}^m} [G_1^{\alpha}(x-\tau,t)f_1(\tau) + G_2^{\alpha}(x-\tau,t)f_2(\tau)]d\tau, \quad (33)$$

where $G_1^{\alpha}(x,t)$ *is given by* (31) *and*

$$G_2^{\alpha}(x,t) = \frac{2^{-m}|x|^{\frac{2-m}{2}}}{\lambda^{1+\frac{m}{2}}\pi^{\frac{m-1}{2}}} t^{-2-\frac{\alpha(m-2)}{4}}$$

$$\times H^{2,0}_{2,2}\left[\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}\middle|\begin{matrix}\left(\frac{m}{4},\frac{1}{2}\right), & \left(-1-\frac{\alpha(m-2)}{4},\frac{\alpha}{2}\right)\\ \left(\frac{m}{2}-1,1\right) & \left(\frac{1}{2}-\frac{m}{4},\frac{1}{2}\right)\end{matrix}\right] \quad (34)$$

provided that the integral in (33) *converges.*

For $m = 1$, the $H^{2,0}_{2,2}$-functions in (31) and (34) reduce to the special Wright function [8, 18.1(21)]

$$\varphi(a,b;z) = \sum_{j=0}^{\infty}\frac{z^j}{j!\Gamma(aj+b)}, \quad a > 0,\ b > 0,\ z \in C, \quad (35)$$

and Theorems 3 and 4 imply the corresponding results for Eq. (5).

Corollary 3. *If* $0 < \alpha \le 1$ *and* $\lambda > 0$, *then the Cauchy-type problem*

$$\begin{aligned}(D^{\alpha}_{0+,t}u)(x,t) &= \lambda^2\frac{\partial^2 u(x,t)}{\partial x^2},\\ (D^{\alpha-1}_{0+,t}u)(x,0+) &= f(x), \quad x \in R,\ t > 0\end{aligned} \quad (36)$$

is solvable, and its solution is given by (30) *with* $m = 1$, *where*

$$G_1^{\alpha}(x,t) = \frac{1}{2\lambda}t^{\frac{\alpha}{2}-1}\varphi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}\right). \quad (37)$$

Corollary 4. *If* $1 < \alpha < 2$ *and* $\lambda > 0$, *then the Cauchy-type problem for Eq.* (5) *with initial conditions* (32) (*where* $m = 1$) *is solvable, and its solution is given by* (33) *with* $m = 1$, *where* $G_1^{\alpha}(x,t)$ *is defined by* (37) *and*

$$G_2^{\alpha}(x,t) = \frac{1}{2\lambda}t^{\frac{\alpha}{2}-2}\varphi\left(-\frac{\alpha}{2},\frac{\alpha}{2}-1;-\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}\right). \quad (38)$$

For $\alpha = 1$, Theorem 3 implies the known solution

$$\begin{aligned}u(x,t) &= \int_{\mathbf{R}^m} G(x-\tau,t)f(\tau)d\tau,\\ G(x,t) &= \frac{1}{(2\lambda\sqrt{\pi})^m}t^{-\frac{m}{2}}e^{-\frac{|x|}{4\lambda^2 t}}\end{aligned} \quad (39)$$

to the Cauchy problem for heat equation (3):

$$\frac{\partial u(x,t)}{\partial t} = \lambda^2(\Delta_x u)(x,t), \quad u(x,0) = f(x),$$
$$x \in \mathbf{R}^m, \quad t > 0. \tag{40}$$

In particular, for $m = 1$, the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = f(x),$$
$$x \in \mathbf{R}, \quad t > 0 \tag{41}$$

has the solution

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\tau,t) f(\tau) d\tau,$$
$$G(x,t) = \frac{1}{2\lambda\sqrt{\pi}} t^{-\frac{1}{2}} e^{-\frac{|x|}{4\lambda^2 t}}. \tag{42}$$

ACKNOWLEDGMENTS

The authors thank the participants of the seminar of Academician S.M. Nikol'skii, Corresponding Member of the Russian Academy of Sciences O.V. Besov, and Corresponding Member of the Russian Academy of L.D. Kudryavtsev at the function theory division of Steklov Institute of Mathematics, Russian Academy of Sciences, for useful discussions.

This work was supported by the Foundation for Fundamental Research of Republic Belarus (project no. F05MS-050).

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